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Likelihood-based inference with singular information matrix

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We consider likelihood-based asymptotic inference for a p-dimensional parameter θ of an identifiable parametric model with singular information matrix of rank $p-1$ at $\theta = \theta^*$ and likelihood differentiable up to a specific order. We derive the asymptotic distribution of the likelihood ratio test statistics for the simple null hypothesis that $\theta = \theta^*$ and of the maximum likelihood estimator (MLE) of θ when $\theta = \theta^*$. We show that there exists a reparametrization such that the MLE of the last $p-1$ components of θ converges at rate $O_p\left(n^{-1/2}\right)$. For the first component θ_1 of θ the rate of convergence depends on the order s of the first non-zero partial derivative of the log-likelihood with respect to θ_1 evaluated at θ^*. When s is odd the rate of convergence of the MLE of θ_1 is $O_p\left(n^{-1/2s}\right)$. When s is even, the rate of convergence of the MLE of $|\theta_1 - \theta_1^*|$ is $O_p\left(n^{-1/2s}\right)$ and, moreover, the asymptotic distribution of the sign of the MLE of $\theta_1 - \theta_1^*$ is non-standard. When $p = 1$ it is determined by the sign of the sum of the residuals from the population least-squares regression of the $(s+1)$th derivative of the individual contributions to the log-likelihood on their derivatives of order s. For $p > 1$, it is determined by a linear combination of the sum of residuals of a multivariate population least-squares regression involving partial and mixed derivatives of the log-likelihood of a specific order. Thus although the MLE of $|\theta_1 - \theta_1^*|$ has a uniform rate of convergence of $O_p\left(n^{-1/2s}\right)$, the uniform convergence rate for the MLE of θ_1 in suitable shrinking neighbourhoods of θ_1^* is only $O_p\left(n^{-1/(2s+2)}\right)$.

Keywords: constraint estimation; identifiability; likelihood ratio test; non-ignorable non-response; reparametrization; rate of convergence